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SINKERS OF THE TITANICS

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On September 5, 1984, Professor Wilfrid Keller of the University of Hamburg proved that the 7,067-digit number $5 \cdot 2^{23473} + 1$ is prime. It is the largest non-Mersenne prime known, and it is larger than all known Mersenne primes except for three of them. In 1971, Bryant Tuckerman discovered a 6,0002-digit Mersenne prime [1], and, although five larger Mersenne primes have been found since then, Professor Keller's discovery is the only known non-Mersenne prime that is larger than Tuckerman's prime, but it may not enjoy this unique status very long.

On September 4, one day before Keller's find, Harvey Dubner of Dubner Computer Systems in Fort Lee, New Jersey, believed that he had found the largest known non-Mersenne prime when he observed that $4974 \cdot 10^{4796} + 1$ is prime. It is a 4,800-digit number, writed as 4974 followed by 4,795 zeros, and a 1. Actually, Professor Keller had found a 5,573-digit prime in mid-August, but news of the discovery had not been reported abroad immediately. *Sic transit gloria!*

In a recent paper [2], primes that are written with a thousand or more decimal digits were called *titanic primes*. A total of 319 of them were listed in an accompanying table; six more were appended before publication. Before 1979, only seven titanic primes were known, and all of them were Mersenne primes.

In the past year, there has been a barrage of newly found primes that have sunk most of the listed ones deeper down the nether regions of the table. By January 1, 1985, the number of known titanic primes had increased to 581, and the number with more than 2,000 digits had gone from 73 to 170.

The *sinkers of the titanics* are some of the same people who had made many of the earlier discoveries. Besides Keller and Dubner, they are Hiromi Suyama of Karatsu, Japan, and Professor A. Oliver L. Atkin of the University of Illinois at Chicago Circle. All of them use techniques that are modifications and improvements of methods which have been described in recent journals. Because they constantly make changes and are eager to see what kinds of results they can obtain, their production methods and the quality of their output are more advanced that what appears in current publications. It is by an interchange of correspondence with these men that I have been fortunate enough to serve somewhat as a receiving and disbursing agent for news of their accomplishments.

It is quite appropriate to compare them and other producers of large primes with those men who once discovered and explored new lands and pioneered in establishing new settlements. They exemplify adventurousness, imagination, solid background knowledge, perseverance, and patience. Despite this analogy, there is no stereotype, each individual achieving significantly in his own way.

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Professor Atkin has been working with Neil Rickert, generating primes that are mostly of the form $k \cdot 2^n + 1$, in their successful quest to obtain large twin and non-Mersenne primes, as well as factors of Fermat numbers. Theirs was for a while the largest known non-Mersenne prime. Their 2,003-digit twin primes $520995090 \cdot 2^{6624} \pm 1$ are the largest twins known. They have also found titanic twin primes $219649815 \cdot 2^{4481} \pm 1$ (1,358 digits) and $256200045 \cdot 2^{3426} \pm 1$ (1,040 digits). A large number of primes with more than 2,000 digits were found by them. Professor Atkin recently wrote, "We have an IBM 3081D now, bigger than before, but without any of the special number-crunching features of the Cray or the British DAP machine. We are inclined to go, when we have time, for some prime triplets $(b, b+2, b+6)$ since there is some skill and knowledge involved in proving all three prime, and one is less handicapped by a smaller machine.

Most of Harvey Dubner's number theoretic computations have been with numbers related to repunits (numbers $R(n)$ written as strings of n 1's) and powers of 10. He supplied many large primitive divisors of repunits for tabulations maintained by this writer as well as by Professor Sam Wagstaff of the "Cunningham Project." He has extended the work of H.C. Williams of the University of Manitoba so that it can now be said that all repunits above $R(1031)$ and below $R(6197)$ are composite. Recently Professor Atkin contributed to Dubner's discovery of large twin primes by verifying that one of each of two twin pairs was prime.

Professor Williams generated large primes containing long repdigit strings, including some primes with long repunit strings [3]. These are among the special types of large primes that Rudolf Ondrejka collects. Spurred on by Ondrejka, who writes about these fascinating numbers, Dubner has produced many of them. The reader may verify the interesting fact that Dubner's primes in Table 1, designated as palidromes are indeed true palindromes. Most of Dubner's titanic primes contain large repdigits. His 2,188-digit $872! + 1$ and 4,042-digit $1477! + 1$ and numbers of the form $2 \cdot 3 \cdot 5 \cdot 7 \ldots p + 1$ (i.e., 1 more than the product of the ascending primes sequence), where $p = 4787, 4547,$ and 3229 are exceptions. He has found that numbers of this form are composite when p is greater than 4787 and less than 10133. One other titanic prime that he found when Ondrejka expressed interest in that type is the 1,201-digit number written as seven successive strins of 12345667890 followed by 1,1130 zeroes and a 1. Since then, he has discovered even larger pandigital numbers, the largest of which are shown in Table 1.

Although the computer that Dubner uses is based on an INTEL 8080 microprocessor similar to common home computers, it has been so modified by him and his son that he feels that when used for number theory, it "is within a factor (in speed) of five or ten of a multi-million dollar Cray computer. It appears to be most economical, since its total hardware cost of operation, including amortization, is about \$10.00 per day." He writes that "I can find about two 1,000-digit primes per hour. I discovered the 4,074-digit prime in about 12 hours. It took about six days to discover the 4,800-digit prime."

Hiromi Suyama is a diligent collector and producer of large primes who has researched and written extensively on the subject of factors of Fermat numbers. He writes, "My computers were an 8-bit microprocessor Z-80 (2MHz) and an 8-bit microprocessor MC6809(1MHz). I am now using the MC6809 and a 16-bit microprocessor iAPX 86/20 (=8086 + co-processor 8087, about 4.9 MHz) which is a CPU in a home computer PC-9801 (like IBM-PC). It took 1 hour, 8 minutes and 11 seconds to prove that the 1,694-digit number $53 \cdot 2^{5621} + 1$ is prime."

Table 1. The Largest Known Primes

$$A(K,N) = K^\star(2^{\star\star}N) + 1 \qquad B(K,N) = K^\star(2^{\star\star}N) - 1$$
$$E(K,N) = (K^{\star\star}2)^\star(2^{\star\star}N) + 1 \qquad F(K,N) = K^{\star\star}4)^\star(2^{\star\star}N) + 1$$
$$G(K,N) = (K^\star(10^{\star\star}N) + 1 \qquad R(N) = (10^{\star\star}N - 1)/9$$
$$P(N,K,M) = (10^{\star\star}N + K)^\star(10^{\star\star}M) + 1$$

No.	Prime	Digits	Discoverer[a]	Year	Special Type
1	B(1,132049)	39751	S	1983	Mersenne
2	B(1,86243)	25962	S	1982	Mersenne
3	B(1,44497)	13395	SN	1979	Mersenne
4	A(5,23473)	7067	K	1984	
5	B(1,23209)	6987	N	1979	Mersenne
6	B(1,21701	6533	NN	1978	Mersenne
7	B(1,19937)	6002	T	1971	Mersenne
8	A(18496,18496)	5573	K	1984	Cullen
9	G(7113,4897)	4901	D	1984	
10	G(279,4898)	4901	D	1984	
11	G(4974,4796)	4800	D	1984	
12	A(7,15494	4666	K	1984	
13	G(138,4071)	4074	D	1984	
14	A(7,13496)	4064	K	1984	
15	1477! + 1	4042	D	1984	"Factorial"
16	A(15450435,13281)	4006	AR	1983	
17	A(4549545,13281)	4005	AR	1983	
18	A(5,13165)	3964	K	1979	
19	E(6486,12674)	3823	AR	1983	
20	B(3,12676)	3817	BB	1979	
21	A(139,12614)	3800	K	1979	
22	B(9,12495)	3763	BR	1981	
23	B(12379,12379)	3731	K	1984	Cullen
24	G(8166,3610)	3614	D	1984	
25	A(19,11890)	3581	K	1980	
26	G(378,3510)	3513	D	1984	
27	B(9,11547)	3477	BB	1979	
28	G(639,3410)	3414	D	1984	
29	A(70195125,11202)	3380	AR	1980	
30	B(1,11213)	3376	G	1963	Mersenne
31	G(3936,3310)	3314	D	1984	

[a] Key to Discoverers: AR = A.Oliver L.Atkin, Neil W.Rickert; BB = Walter Borho, Jurgen Buhl; BR = Walter Borho, R.Reckow; CW = G.V.Cormack, Hugh C.Williams; D = Harvey Dubner; G = Donald B.Gillies; K = Wilfred Keller; N = Curt L.Noll; NN = Curt L.Noll, Laura A.Nickel; S = David Slowinski; SN = David Slowinski, Harry L.Nelson; SU = Hiromi Suymama; T = Bryant Tuckerman.

Table 1. (Cont'd.)

No.	Prime	Digits	Discoverer[a]	Year	Special Type
32	G(2439,3310)	3314	D	1984	
33	G(1053,3310)	3314	D	1984	
34	A(111,10883)	3279	K	1981	
35	G(6*R(1611),1612)	3223	D	1984	
36	G(2373,3210)	3214	D	1984	
37	G(357,3100)	3103	D	1984	
38	G(3441,3080)	3084	D	1984	
39	G(6834,3070)	3074	D	1984	
40	G(4419,3070)	3074	D	1984	
41	G(4095,3070)	3074	D	1984	
42	A(167,10183)	3068	K	1979	
43	A(11,10179)	3066	K	1979	
44	G(1965,3020)	3024	D	1984	
45	F(6952,9952)	3012	AR	1983	
46	F(5555,9952)	3011	AR	1983	
47	F(5213,9952)	3011	AR	1983	
48	F(4682,9952)	3011	AR	1983	
49	F(4638,9952)	3011	AR	1983	
50	F(3950,9952)	3011	AR	1983	
51	F(2081,9952)	3010	AR	1983	
52	A(103858755,9952)	3004	AR	1980	
53	A(31336305,9921)	2995	AR	1980	
54	B(1,9941)	2995	G	1963	Mersenne
55	E(694,9920)	2992	AR	1980	
56	P(1488,3,1488)	2977	D	1984	Palindrome
57	A(2897,9715)	2928	K	1979	
58	B(1,9689)	2917	G	1963	Mersenne
59	G(2*R(1439),1440)	2879	D	1984	
60	B(9531,9531)	2874	K	1984	Cullen
61	A(25,9522)	2868	K	1983	
62	A(19,9450)	2847	K	1983	
63	A(9,9431)	2840	K	1983	
64	P(1419,5,1419)	2939	D	1984	Palindrome
65	P(1375,999,1373)	2749	D	1984	Palindrome
66	A(31,9096)	2740	K	1983	
37	A(65057,8899)	2684	K	1983	

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Table 1. (Cont'd.)

No.	Prime	Digits	Discoverer[a]	Year	Special Type
68	P(1335,6,1335)	2671	D	1984	Palindrome
69	G(6,2629)	2630	D	1984	
70	G(3,2620)	2621	D	1984	
71	P(1305,23456789198765432,1289)	2595	D	1984	Pandigital Palindrome
72	A(41,9411)	2534	K	1983	
73	A(14899,8234)	2483	K	1983	
74	P(1242,23456789198765432,1226)	2469	D	1984	Pandigital Palindrome
75	B(9,8007)	2412	BB	1980	
76	A(19,7998)	2409	K	1983	
77	A(9,7967)	2400	AR	1979	
78	B(9,7939)	2391	BB	1980	
79	A(29,7927)	2388	CW	1979	
80	A(271,7780)	2345	K	1983	
81	B(7755,7755)	2339	K	1984	Cullen
82	A(27,7639)	2301	CW	1979	
83	A(41,7607)	2292	K	1983	
84	A(39,7583)	2285	K	1983	
85	B(3,7559)	2276	BB	1980	
86	A(19,7498)	2259	K	1983	
87	A(49,7446)	2244	K	1983	
88	A(15,7392)	2227	K	1983	
89	P(1118,56789123432198765,1102)	2221	D	1984	Pandigital Palindrome
90	P(1110,88088,1106)	2217	D	1984	Palindrome
91	A(17,7311)	2203	CW	1979	
92	872! + 1	2188	D	1983	"Factorial"
93	G(R(820),1506)	2112	D	1984	
94	A(15,7050)	2124	K	1983	
95	G(5*R(606),1506)	2112	D	1984	
96	A(21,6981)	2103	K	1983	
97	A(9,6937)	2090	K	1983	
98	A(45,6923)	2086	K	1983	
99	A(73252,6889	2079	K	1983	
100	A(19,6838)	2060	K	1983	
101	A(15,6804)	2050	K	1983	
102	2*3*6*7* ...* 4787 + 1	2038	D	1984	"Prime-Factorial"
103	A(21,6712)	2022	K	1983	

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Table 1. (Cont'd.)

No.	Prime	Digits	Discoverer[a]	Year	Special Type
104	G(6,1919)*12*R(995) + 1	2015	D	1984	
105	A(91,6668)	2010	SU	1984	
106	P(1003,818,1001)	2005	D	1984	Palindrome
107	A(524477415,6624)	2003	AR	1984	
108	A(520995090,6624)	2003	AR	1984	Twin
109	B(520995090,6624)	2003	AR	1984	Twin
110	A(517377630,6624)	2003	AR	1984	

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A great numbr of Keller's titanic primes are of the forms $k\cdot 2^n+1$ and $k(2^n-1)+1$. He has factored Fermat numbers and found large twin primes. He writes, "Investigating the so-called Cullen numbers $C_n = n \cdot 2^n + 1$ and $W_n = n \cdot 2^n - 1$, I established that C_n is prime for $n = 4713, 5795, 6611$ (previously only $n = 141$ was known to give a prime), and W_n is prime for n =2,3,6,30,75,81,115, 249,362,384,462,512,751,882,5312,7755,9531 (including some previously known primes, like the Mersenne number $m_{521} = W_{512}$." His work was done on a TELEFUNKEN TR440 computer and a SIEMENS 7.882 computer, and the methods and techniques that he used until 1983 were described by him in *Mathematics of Computation* [4].

Table 1 shows the 110 largest known titanic primes, updated to January 1, 1985.

Added in Proof - and updating to April 1, 1985.

Harvey Dubner discovered a new largest non-Mersenne prime! It is the 7,094-digit number $6006\times 10^{7090}+1$. It required two days of running time on his computer. He also found a few probable twin primes larger than the listed known twins. While doing so, he generated more primes with more than 2,000 digits. As of April 1, the total number of titanic primes is 652. Dubner's present largest "pandigital prime" is the 3,284-digit number

$$1_{111}2_{111}3_{111}4_{111}5_{111}6_{111}7_{111}8_{111}9_{111} \times 10^{2285} + 1,$$

using a chemical-type notation.

While proving that the numbers tested are all composite, he has extended the search for additional primes among numbers of the form $n!+1$ to $n = 2043$; among numbers of the form $2 \times 3 \times 5 \times \times 7 \cdots \times p + 1$ to $p = 11213$; and among repunits to $R(6828)$.

No new Mersenne primes have been found since 1983, and no new prime repunits have been verified in almost a decade since $R(317)$ was discovered. Efforts are still being made to show that probable prime $R(1031)$ is prime. The prime that is shown in Table 1 as the 110th largest is now the 147th largest known prime.

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